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Conformal Lie superalgebras and moduli spaces

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Abstract

A conformal Lie superalgebra is a superextension of the centerless Virasoro algebra W—the Lie algebra of complex vector fields on the circle. The algebras of Ramond and Neveu-Schwarz are not the only examples of such superalgebras. All known superconformal algebras can be obtained as comlexifications of Lie superalgebras of vector fields on a supercircle with an additional structure. For every such superalgebra $\mathcal{G}$ a class of geometric objects—complex $\mathcal{G}$-supercurves—is defined. For the superalgebras of Neveu-Schwarz and Ramond they are super Riemann surfaces with punctures of different kinds. We construct moduli superspaces for compact $\mathcal{G}$-supercurves, and show that the superalgebra $\mathcal{G}$ acts infinitesimally on the corresponding moduli space.

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1. Introduction

The recent developments in the theory of quantum strings and two-dimensional conformal field theories connected two seemingly remote branches of mathematics: the theory of infinite dimensional Lie algebras and their representations and the geometry of moduli spaces of the Riemann surfaces and vector bundles on them.

The first corresponds to the Hamiltonian (operator) formalism, and the second arises in the Lagrangian (path integral) picture.

In [M1] Yu. Manin gave some evidence for the existence of a direct mathematical connection between these two theories and conjectured on the possible form it might have.

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The connection has been simultaneously established in [ACKP, BS, K]. It turned out that the Lie algebra W (usually called the Witt algebra or centerless Virasoro algebra) of complex-valued vector fields on the circle acts on the space of triples (C, p, z), where C is a compact Riemann surface, $p \in C$, and z is a local parameter at p.

This action changes the complex structure on C and pushes down to an infinitesimal transitive action of W on the space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. The Virasoro algebra $\hat{W}$, a nontrivial central extension of W, acts in the total space of the determinant bundle over $\mathcal{M}_g$. This is due to isomorphism between the second cohomology group of $\mathcal{M}_g$ and the second cohomology of the adjoint representation of W. In terms of this action one can write the differential equations for the Polyakov–Mumford measure on the moduli space, for the correlator functions, etc.

In the theory of fermionic strings the operator formalism and Polyakov's path integral relate, respectively, the representation theory of the Neveu–Schwarz superalgebra (shortly, NS) and the moduli spaces of $(1|1)$-dimensional supermanifolds with a contact structure (also known as super Riemann surfaces or SUSY-curves). The mathematical counterpart of this relationship has not yet been understood as well as in the bosonic case, but even the obtained results (see [D1, Vo, Ma2, UY]) leave no doubt that everything here is analogous to the non-super situation.

Since the early 70s it has been known that besides the Neveu–Schwarz superalgebra there exist other simple superextensions of the Witt algebra, some of them having nontrivial central extensions similar to the Virasoro algebra.

The problem of classifying such superconformal Lie superalgebras and their central extensions was first addressed by Leites and Feigin [FL]. They found that all known superconformal Lie superalgebras have a natural geometric interpretation. These superalgebras are subalgebras of the Lie superalgebra $W(1|N)$ of Laurent vector fields on the superdisk $U^{1|N} = (U, \mathbb{C}[z, z^{-1}] \otimes \Lambda^*(\mathbb{C}^N)$, where $U = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, or, in other words, of the superalgebra of derivations of $\mathcal{O}_{U^{1|N}} = \mathbb{C}[z^{-1}, z, \theta_1, ..., \theta_N]$. Such subalgebras form several series (cf. Section 2.2).

The list of cocycles and even of the algebras themselves in [FL] is incomplete. The classification was renewed by Kac and van de Leur [KL], who conjectured that the list is now complete.

Representations of superconformal superalgebras other than NS are under active study, but the geometric side of the picture, sketched above for the Virasoro algebra, has not been developed fully. Only quite recently have papers on moduli spaces of $N = 2$ superconformal manifolds started to appear [M4, FR, DRS], and none, so far, has addressed the relationship of these moduli spaces with the $N = 2$ superconformal Lie superalgebra.

The main purpose of this paper is to define a class of geometric objects for each of the known superconformal Lie superalgebras $\mathcal{G}$. These $\mathcal{G}$-superconformal manifolds or supercurves of type $\mathcal{G}$ are related to the corresponding Lie superalgebra in the same way as Riemann surfaces are related to the Virasoro algebra—the superalgebra $\mathcal{G}$ acts on their moduli spaces.

For $\mathcal{G} = K(1|N)$, the standard N-superconformal Neveu–Schwarz superalgebra, our

$\mathcal{G}$-supercurves coincide with the N-superconformal manifolds of [DRS].

The main result of the paper is a theorem on the existence of a complex analytic moduli superspace for each of these objects. The proof is based on the technique of the deformation theory of superanalytic structures, similar to that used in [V1] for construction of the moduli space of SUSY-curves.

For basic notions on supermanifolds and their geometry we refer the reader to [L1, M3].

2. Supercircles

All the known superextensions of the Witt and Virasoro Lie algebras are Lie superalgebras of (complex valued) vector fields preserving certain geometric structures on a supermanifold with the circle S^1 as the underlying space. Therefore, first we will discuss these geometric objects.

The Witt algebra is defined as the complexification of the Lie algebra of real vector fields on the circle. It is, as we now know, the only simple $\mathbb{Z}$-graded complex Lie algebra of finite growth which belongs neither to Kac–Moody type algebras nor to the algebras of vector fields on $\mathbb{C}^n$. This algebra has several superextensions due to the fact that there are several superextensions of the circle.

Definition 2.1. A *supercircle* is a real supermanifold M, such that its underlying manifold M_{rd} is diffeomorphic to S^1.

Any real supermanifold M is split—functions on M are sections of the Grassmann algebra $\Lambda^*(E)$ of a vector bundle over the underlying manifold M_{rd}. Since there exist only two non-equivalent vector bundles of rank n over the circle, we obtain the following description of supercircles.

Proposition 2.2. *For any $N > 0$ there are only two supercircles of dimension $1|N$:*

$$S^{1|N} = (S^1, \Lambda^*(\mathbb{R}^N)) \quad \text{and} \quad S_+^{1|N} = (S^1, \Lambda^*(\mathbb{R}^{N-1} \oplus M)), \tag{1}$$

where M is the Möbius line bundle over the circle.

Remark 2.3. The supercircle $S_+^{1|N}$ is diffeomorphic to the projective superspace $\mathbb{RP}^{1|N}$.

2.1. *Geometric structures on supercircles*

Whereas the circle S^1 has very little room for extra geometric structures, the supercircles $S^{1|N}$ and $S_+^{1|N}$ may be equipped with additional structures and still have a sufficiently large diffeomorphism group.

Let S be a supercircle of dimension $1|N$. Fix on S a coordinate system $(t, \theta) = (t, \theta_1, \ldots, \theta_{N-1}, \theta_N)$, where $t \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $\theta_1, \ldots, \theta_{N-1}$ are sections of trivial line bundles over S^1, and θ_N is a section of the trivial line bundle if $S = S^{1|N}$ and of the Möbius one if $S = S_+^{1|N}$.

We will call an *equipped supercircle* one of the following objects:

(1) S, the supercircle $S^{1|N}$.

(2) S_Δ, the supercircle $S^{1|N}$ with a volume form Δ on it.

(3) $S_{\Delta,\lambda}$, the supercircle $S^{1|N}$ with the volume form $t^\lambda\Delta$.

(4) S_K, the supercircle $S^{1|N}$ with an even contact structure $V \subset T_{S^{1|N}}$—maximally non-integrable distribution of codimension $1|0$.

There exists a coordinate system on $S^{1|N}$ in which the distribution is given as kernel of the 1-form

$$\omega = dt - \sum_{i=1}^{N} \theta_i \, d\theta_i. \tag{2}$$

(5) S_{K+}, the supercircle $S_+^{1|N}$ with an even contact structure.

(6) S_α, the supercircle $S^{1|2}$ with *odd* contact structure given by the 1-form

$$\pi = \theta_1 - t\theta_2 - \theta_1 \, dt. \tag{3}$$

It is not difficult to show that these objects have a large group of automorphisms and that there is essentially one such object in every class.

Proposition 2.4. *Let S be an equipped supercircle. Then*

(i) any orientation preserving diffeomorphism of S^1 can be lifted to an automorphism of S;

(ii) any two such objects S_1 and S_2 of the same type are equivalent.

Supergroups of diffeomorphisms of these objects, the corresponding Lie superalgebras and their complexifications are supergeneralizations of respectively $Diff_+(S^1)$, $Vect(S^1)$ and $W = Vect_{\mathbb{C}}(S^1)$.

2.2. Complexifications

The Witt algebra—the complexification of the Lie algebra of polynomial vector fields on the circle—can be realized as the Lie algebra of Laurent vector fields on the punctured unit disk $U = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$.

The complexifications of Lie superalgebras of polynomial vector fields on equipped supercircles become Lie superalgebras of Laurent polynomial vector fields on superextensions of the punctured disk U. Since all holomorphic bundles over a Stein manifold are trivial, the only $(1|N)$-dimensional complex supermanifold M with $M_{rd} = U$ is $U^{1|N} = (U, \mathbb{C}[t, t^{-1}] \otimes \Lambda^*[\theta_1, \ldots, \theta_N])$.

The following superdisks with additional structures will be considered as "complexifications" of the corresponding equipped supercircles from 2.1.

(1) The superdisk $U^{1|N}$.

(2) U_Δ, the superdisk $U^{1|N}$ with a holomorphic volume form Δ (= nowhere vanishing section of the Berezinian line bundle) with constant coefficients.

(3) $U_{\Delta,\lambda}$, the superdisk $U^{1|N}$ with the volume form $\Delta_\lambda = t^\lambda \Delta$. [2]

(4) U_K, the superdisk $U^{1|N}$ with a holomorphic contact structure given by the 1-form ω (eq. 2). This contact structure extends to a contact structure on the whole non-punctured superdisk $\bar{U}^{1|N}$.

(5) U_{K+}, the superdisk $U^{1|N}$ with a contact structure given by the 1-form

$$\omega_+ = dt - \sum_{i=1}^{N-1} \theta_i \, d\theta_i - t\theta_N \, d\theta_N. \tag{4}$$

This form does not define a contact structure on $\bar{U}^{1|N}$.

(6) U_α, the superdisk $U^{1|2}$ with the odd contact structure given by the 1-form (3).

Proposition 2.5 ([FL]). *Let $\mathcal{G}$ be the Lie superalgebra of Laurent polynomial vector fields on $U^{1|N}$ preserving one of the structures (1)–(6) above. Then $\mathcal{G}$ is isomorphic to the complexification of the Lie superalgebra of the supergroup of polynomial automorphisms of the corresponding equipped supercircle.*

3. Superconformal algebras

Definition 3.1 ([KL]). A complex $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G} = \bigoplus \mathcal{G}_j$ is called *superconformal* if

(i) $\mathcal{G}$ is simple;
(ii) $\mathcal{G}$ contains the Witt algebra W as a subalgebra;
(iii) $\mathcal{G}$ has growth 1.

The last condition means that $\dim \mathcal{G}_j < Cj$, where C is a constant independent of j.

The Lie superalgebra $W(1|N)$ of all Laurent polynomial vector fields on the superdisk $U^{1|N}$ can be defined as the superalgebra of all derivations of

$$\mathcal{O}_N = \mathbb{C}[t, t^{-1}] \otimes \Lambda^*[\theta_1, \ldots, \theta_N], \tag{5}$$

where $\Lambda * [\theta_1, \ldots, \theta_N]$ is the Grassmann algebra with N generators .

Every element $X \in W(1|N)$ can be expressed as

$$X = f_0 \frac{\partial}{\partial t} + \sum_{i=1}^{N} f_i \frac{\partial}{\partial \theta_i}, \quad f_i \in \mathcal{O}_N. \tag{6}$$

The vector field X preserves the constant volume form Δ on $U^{1|N}$ if and only if its *divergence*

[2] If $\lambda \notin \mathbb{Z}$, then the form Δ is no longer a single-valued one. This is not a problem by the 19th century standards, but now an explanation is required. Proper translation into modern terminology can be done by using the language of branched covering spaces (cf. [V4]).

$$\operatorname{div} X = \frac{\partial f_0}{\partial t} + \sum_{i=1}^{N} (-1)^{\bar{f}_i} \frac{\partial f_i}{\partial \theta_i} \tag{7}$$

is zero, where $\bar{f} \in \mathbb{Z}_2$ is the *parity* of $f \in \mathcal{O}_N$.

The *Lie derivative* of the 1-form $\alpha = a_0\, dt + \sum a_i\, d\theta_i$ with respect to X is defined as

$$L_X(\alpha) = \left(f_0 \frac{\partial a_0}{\partial t} + \sum_{i,j=1}^{N} f_i \frac{\partial a_0}{\partial \theta_j} + (-1)^{\bar{X}\bar{\omega}} (a_0 \frac{\partial f_0}{\partial t} + \sum_{i,j=1}^{N} a_i \frac{\partial f_j}{\partial t}) \right) dt$$
$$+ \sum_{k=1}^{N} \left(f_0 \frac{\partial a_k}{\partial t} + \sum_{i,j=1}^{N} f_i \frac{\partial a_k}{\partial \theta_j} + (-1)^{\bar{X}\bar{\omega}} (a_0 \frac{\partial f_0}{\partial \theta_k} + \sum_{i,j=1}^{N} a_i \frac{\partial f_j}{\partial \theta_k}) \right) d\theta_k. \tag{8}$$

Now the Lie superalgebras from Proposition 2.4 can be described as the following $\mathbb{Z}$-graded subalgebras of $W(1|N)$:

(1) $W(1|N) = \operatorname{Der}(\mathcal{O}_N)$.

(2) $S(1|N) = \{X \in W(1|N) \mid \operatorname{div} X = 0\}$, the divergence free superalgebra.

(3) $S(1|N,\lambda) = \{X \in W(1|N) \mid \operatorname{div}(t^\lambda X) = 0\}$, the deformed divergence free superalgebra. This algebra is well-defined for all $\lambda \in \mathbb{C}$ even though the corresponding volume form $t^\lambda \Delta$ on $U^{1|N}$ makes sense only for real λ.

(4) $K(1|N) = \{X \in W(1|N) \mid L_X\omega = f\omega$ for some $f \in \mathcal{O}_N\}$, the contact superalgebra (ω is the contact 1-form (2) on $U^{1|N}$).

(5) $K(1|N)_+ = \{X \in W(1|N) \mid L_X\omega_+ = f\omega_+$ for some $f \in \mathcal{O}_N\}$, the twisted contact superalgebra (ω_+ is the contact 1-form (4) on $U^{1|N}$).

(6) $M(1|2) = \{X \in W(1|N) \mid L_X\pi = f\pi$ for some $f \in \mathcal{O}_N\}$, the odd contact superalgebra (π is the odd contact 1-form (3) on $U^{1|N}$).

These algebras provide all the known examples of superconformal Lie superalgebras.

Theorem 3.2 ([FL,KL]).

(a) The superalgebras $W(1|N)$, $S(1|N,\lambda)$, $K(1|N)$, $K(1|N)_+$, $M(1|2)$ contain the Witt algebra W as a subalgebra and have growth 1.

(b) The superalgebras $W(1|N)$, $S(1|N,\lambda)$ for $N > 1$ and $\lambda \notin \mathbb{Z}$, $K(1|N)$ for $N \neq 4$, $K(1|N)_+$, $M(1|2)$ are simple and, therefore, superconformal.

(c) For $\mathcal{G} = K(1|4)$ or $S(1|N,\lambda)$ with $\lambda \in \mathbb{Z}$ and $N > 1$ the derived *subsuperalgebra $\mathcal{G}' = [\mathcal{G},\mathcal{G}]$ has codimension 1 in $\mathcal{G}$ and is simple and superconformal.*

(d) The only non-trivial isomorphisms between these superconformal algebras are

$$W(1|1) \simeq K(1|2) \simeq M(1|2) \text{ and } S(1|N,\lambda) \simeq S(1|N,\mu)$$

when $\lambda - \mu \in \mathbb{Z}$.

(e) For a superconformal algebra $\mathcal{G}$ of (b) or (c) a non-trivial central extension exists only if $\mathcal{G}$ is one of the following

$$W(1|1);\ W(1|2);\ S(1|2,\lambda), \lambda \notin \mathbb{Z};\ S'(1|2,\lambda), \lambda \in \mathbb{Z};$$

$K(1|N)$, $N \leq 3$; $K'(1|4)$; $M(1|2)$.

Conjecture 3.3 ([KL]). *Any superconformal algebra is isomorphic to one of the above algebras.*

Remarks 3.4.

(1) A superconformal superalgebra which has a nontrivial central extension is called *distinguished.* These superalgebras are distinguished because only such algebras have non-trivial unitary projective positive energy representations, which is very important for applications in physics.

(2) The Lie algebra $W(1|0)$ is the Witt algebra W. Its nontrivial central extension is the Virasoro algebra $\hat{V}$.

(3) The nontrivial central extensions of Lie superalgebras $K(1|1)$ and $K_+(1|1)$ are called the Neveu–Schwarz and Ramond Lie superalgebras, respectively. They were the first examples of superextensions of the Virasoro algebra.

4. Supercurves corresponding to superconformal algebras

Let us assign to every superconformal Lie superalgebra $\mathcal{G}$ a $\mathcal{G}$-curve, which is a $(1|N)$-dimensional complex supermanifold C with an additional structure. The superdisks 2.2(1)–(6) will serve as local models for the following.

Definition 4.1.

(1) A $W(1|N)$-supercurve is a complex supermanifold C of dimension $(1|N)$ with a finite number of points $p_1, \dots, p_k \in C$ (punctures).

(2) A $S(1|N)$-supercurve is a complex supermanifold C of dimension $(1|N)$ with a holomorphic volume form—a nowhere vanishing section ρ of the Berezinian line bundle Ber_C and a finite number of punctures.

(3) A $S(1|N, \lambda)$-supercurve is a complex supermanifold C of dimension $(1|N)$ with a finite number of punctures $p_1, \dots, p_k \in C$ and a non-vanishing section $\rho \in H^0(C \setminus \{p_1, \dots, p_k\}; \mathrm{Ber}_C)$ which has singularities of the type t^λ at the punctures.

(4) A $K(1|N)$-supercurve is a complex supermanifold C of dimension $(1|N)$ with a holomorphic even contact structure and a finite number of punctures.

(5) A $K(1|N)_+$-supercurve is a complex supermanifold C of dimension $(1|N)$ with a finite number of punctures $p_1, \dots, p_k \in C$ and a distribution $V \subset T_C$ of codimension $1|0$ which gives a contact structure on $C \setminus \{p_1, \dots, p_k \in C\}$, and in a neighbourhood of a puncture is equivalent to the contact structure (4).

(6) A $M(1|2)$-supercurve is a complex supermanifold C of dimension $(1|2)$ with a holomorphic odd contact structure and a finite number of punctures.

Remarks 4.2.

(1) The number of punctures may be zero. In this case $S(1|N;\lambda)$-curves will not be distinguished from $S(1|N)$-curves, and $K(1|N)_+$-curves not from $K(1|N)$-curves.

(2) For $S(1|N;\lambda)$-curves in the case $\lambda \notin \mathbb{Z}$ a rigorous definition requires more care (cf. footnote 2).

4.1. Examples

(1) The first example of a $\mathcal{G}$-supercurve is the corresponding superdisk (2.2). These superdisks are model objects; any other $\mathcal{G}$-supercurve can be built by gluing superdisks.

(2) $K(1|N)$-*supercurves.* For a complex $1|N$-dimensional supermanifold C and a $0|N$-dimensional distribution $V \subset T_C$ the following Frobenius form is well-defined:

$$\phi : V \otimes V \to T_C/V \ : \ (v, w) \mapsto [v, w] \bmod V. \tag{9}$$

The distribution V is a contact structure if ϕ is a non-degenerate form. In this case we can find a local coordinate system $\Theta = (t, \theta_1, \ldots, \theta_N)$, such that the fields $D_{i,\Theta} = \partial/\partial\theta_i + \theta_i\,\partial/\partial t$ diagonalize the form ϕ: $[D_i, D_j] = 2\delta_{ij}\,\partial/\partial t$. In coordinates Θ the distribution $V = \ker(dt - \sum \theta_i\, d\theta_i)$ is spanned locally by the fields D_i.

This structure is preserved by contact coordinate transformations: if Θ transforms to Θ', then $D_{i,\Theta'} = \sum F_{ij}(\Theta) D_{j,\Theta}$. This means that the notions of $K(1|N)$-supercurves and N-super Riemann surfaces of [Ch, DRS] coincide.

(3) $K(1|1)$-*supercurves from the even point of view.* Let C be a $K(1|1)$-supercurve (or SUSY-curve in Manin's notations [M1]). The contact distribution V is $(0|1)$-dimensional, therefore the Frobenius form gives an isomorphism $V \otimes V \simeq T_C/V$. Reducing to the underlying Riemann surface C_{rd}, we get isomorphisms $V_{rd} \otimes V_{rd} \simeq T_{C_{rd}}$ and $V^*_{rd} \otimes V^*_{rd} \simeq \Omega_{C_{rd}}$, where $\Omega_{C_{rd}}$ is the canonical bundle of C_{rd}. Therefore, the line bundle V_{rd} is a theta characteristic (a square root of the canonical bundle).

Now, given a theta characteristic E on a Riemann surface X with an isomorphism $\pi : E \otimes E \to \Omega_X$, we can construct a $K(1|1)$-supercurve. Consider a $(1|1)$-dimensional supermanifold $\mathcal{X} = (X, \Lambda^*(E))$ and define a contact structure V on $\mathcal{X}$ as given locally by the vector field $D = \partial/\partial\zeta + \zeta\,\partial/\partial z$, where z is a local parameter on X and ζ is a section of E, such that $\pi(\zeta \otimes \zeta) = dz$.

(4) $K(1|2)$-*supercurves versus* $W(1|1)$-*supercurves.* Locally every $K(1|N)$-supercurve C has N-subbundles of rank $0|1$ in which the Frobenius form ϕ is diagonal. If there exists a global choice of such subbundles, we call the $K(1|N)$-supercurve C orientable (untwisted in the terminology of [Ch]).

P. Deligne noticed [D2] that there exists a correspondence between oriented $K(1|2)$-supercurves and $W(1|1)$-supercurves. If we have an oriented $K(1|2)$-supercurve C, we may find two subbundles E_1 and E_2 in V which are isotropic with respect to ϕ. The supermanifold $(C_{rd}, \Lambda^*(E_{1,rd}))$ is the $W(1|1)$-supercurve, which corresponds to C. The inverse construction maps a $1|1$-supermanifold X to the relative Grassmannian $Gr(0|1, T_X)$, which has a natural structure of oriented $K(1|2)$-supermanifold.

This correspondence reflects the isomorphism $K(1|2) \simeq W(1|1)$ of superconformal algebras. The other isomorphism $M(1|2) \simeq W(1|1)$ also has a geometric counterpart.

5. Moduli spaces

A local complex analytic structure on the superspace of g-curves is introduced with the help of deformation theory [V1] as follows.

Theorem 5.1. *Let $\mathcal{G}$ be one of the superconformal Lie superalgebras of Theorem 3.2(a), (b). If $\mathcal{G} \neq S(1|N;\lambda)$ with $\lambda \notin \mathbb{Q}$ then for any compact $\mathcal{G}$-curve C there exists a versal deformation $\pi : \mathcal{C} \to B$ whose base B is a finite-dimensional complex supermanifold.*

Proof. Let us first consider a supercurve C of W-type without punctures. Then C is just a compact complex supermanifold without any structures. The existence of a versal deformation in this case was proved in [V2]. The dimension of the base B of the deformation is equal to $\dim H^1(C;T_C)$, and its smoothness follows from the vanishing of $H^2(C;T_C)$.

For $S(1|N)$-supercurves we have to deform a section $\Delta \in H^0(C;\mathrm{Ber})$ as well as the supermanifold C. This can be done by applying the theorem on deformations of cohomology classes of coherent analytic sheaves [V3].

For a supercurve of type $K(1|N)$ or $M(1|2)$ the extra structure on C is a contact distribution—a subbundle F of the tangent bundle T_C. By analogy with the $K(1|1)$ case considered in [V1], we construct first a versal deformation of C, then a versal deformation of F as a holomorphic bundle on C, then a simultaneous deformation of C and F, and at last, a deformation of the embedding $f : F \hookrightarrow T_C$ considered as an element of $H^0(S; Hom(F,T_C))$. The resulting deformation will produce a subbundle $\mathcal{F}$ in the tangent bundle of the deformation $\mathcal{C}$ of the supermanifold C. Since being a contact structure is an open condition, $\mathcal{F}$ is a contact structure in a neighbourhood of F.

Punctures $p_1,\dots,p_k$ and degeneracies of the volume forms and contact structures at them can be dealt with in a similar way. Punctures are deformed as submanifolds in C, and singular volume forms and contact structures as sections of coherent analytic sheaves on C.

The details of the proof will appear in [V4].

5.1. Examples

(1) As we know from the discussion in 4.1 a $K(1|1)$-supercurve C is just a theta-characteristic on C_{rd}. If C_{rd} has genus g then there are precisely 4^g theta-characteristics on C_{rd}. Therefore, the even part of the moduli for C has the same dimension as the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. But the odd part of the space of

deformations of C has a non-trivial structure. If $g > 1$, the dimension of the moduli superspace is $3g - 3|2g - 2$.

The moduli of $K(1|1)$-curves have been studied in numerous mathematical and physical papers (cf. for example, [M1] and [BFS]). A detailed description of the complex structure on this superspace can be found in [LR] and [CR]. Deligne [D1] constructed a compactification of this superspace.

(2) A $W(1|1)$-supercurve C from the even point of view is just a Riemann surface C_{rd} with a holomorphic line bundle E. The even part of the moduli space is, therefore, the total space of the universal Picard bundle over $\mathcal{M}_g$. However, even this classical object gets a nontrivial superextension. If $g > 1$ and the bundle E is not trivial, then the dimension of the deformation of C is $4g - 3|3g - 4$, and it is equal to $4g - 3|3g - 3$ if E is trivial.

(3) The isomorphisms 3.2(b) $K(1|2) \simeq W(1|1) \simeq M(1|2)$ suggest relations between the corresponding moduli spaces. As we have seen in 4.1, such relations exist, though in general the moduli spaces are not identical. For example, the moduli space of $W(1|1)$-supercurves is approximately one half of the whole $K(1|2)$-moduli space.

The superspace of moduli of $K(1|2)$-curves is considered in [FR] and [DRS].

6. Action of $\mathcal{G}$ on moduli

The moduli space $\mathcal{M}_g$ of compact Riemann surfaces is an infinitesimal homogeneous space for the Witt algebra (cf. [ACKP, BS, K]). Similar relation exists between a superconformal algebra $\mathcal{G}$ and the moduli of $\mathcal{G}$-supercurves.

6.1

Let $\mathcal{G}$ be one of the algebras 3.2(a) with the exception of $S(1|N;\lambda)$ for $\lambda \notin \mathbb{Q}$. Consider a completion $\bar{\mathcal{G}}$ of $\mathcal{G}$ which consists of all formal Laurent vector fields, preserving the structure on the $\mathcal{G}$-superdisk. Take a $\mathcal{G}$-supercurve C with punctures $p_1, \dots, p_k$. We call a local coordinate system $Z = (z, \theta_1, \dots, \theta_k)$ around one of the punctures *canonical* if the restriction of the $\mathcal{G}$-structure on C to $U_Z = \{p \in C_{rd} \mid 0 < |z| < 1\}$ in these coordinates coincides with the standard one of 2.2.

Consider the set $\hat{\mathcal{M}}_{\mathcal{G},k}$ of equivalent classes of objects

$$\mathcal{C} = (C; (p_1, Z_1); \dots; (p_k, Z_k)), \tag{10}$$

where C is a compact $\mathcal{G}$-curve, $p_1, \dots, p_k$ are the punctures, and Z_i is a canonical coordinate system at p_i.

$\hat{\mathcal{M}}_{\mathcal{G},k}$ has a natural structure of an infinite-dimensional (formal) complex supermanifold. In other words,

$$\hat{\mathcal{M}}_{\mathcal{G},k} = \varprojlim \hat{\mathcal{M}}^d_{\mathcal{G},k},$$

where $\hat{\mathcal{M}}^d_{\mathcal{G},k}$ is the collection of objects of the form (10) with Z_i replaced by their d-jets (that is, Z_i is taken modulo I_i^{d+1}, where I_i is the defining ideal of the point p_i).

Denote by $\mathcal{T}_C$ the sheaf of holomorphic vector fields which preserve the $\mathcal{G}$-structure on C. Then we have $H^0(U_{Z_i};\mathcal{T}_C) = \bar{\mathcal{G}}$. The open subsets $\bar{U}_{Z_1},\dots,\bar{U}_{Z_k}$ and $C\setminus\{p_1,\dots,p_k\}$ form a Stein covering of C, therefore we may compute the cohomology of coherent sheaves using this covering. The Kodaira–Spencer map

$$T_C\hat{\mathcal{M}}^d_{\mathcal{G},k} \longrightarrow H^1(C;\mathcal{T}_C\otimes\prod_i (I_i)^{d+1})$$

is an isomorphism for $d\geq 1$. As $d\to\infty$, we get a short exact sequence for computing $T_C\hat{\mathcal{M}}^d_{\mathcal{G},k}$:

$$0\to H^0(C\setminus\{p_1,\dots,p_k\};\mathcal{T}_C)\to H^0(\coprod_i U_{Z_i};\mathcal{T}_C)\to T_C\hat{\mathcal{M}}^d_{\mathcal{G},k}\to 0.$$

Finally, $T_C\hat{\mathcal{M}}^d_{\bar{\mathcal{G}},k} = \mathcal{G}^k/L$, where L consists of k-tuples $(g_1,\dots,g_k)$, $g_i\in\bar{\mathcal{G}}_i = H^0(U_{Z_i};\mathcal{T}_C)$, which can be extended to a vector field $v\in H^0(C\setminus\{p_1,\dots,p_k\};\mathcal{T}_C)$.

Thus we obtain an epimorphism $\rho:\bar{\mathcal{G}}^k\to Vect(\hat{\mathcal{M}}_{\mathcal{G},k})$. We can check that ρ is, actually, a homomorphism of Lie superalgebras, and therefore, it gives the desired action.

Theorem 6.1. *The Lie superalgebra $\bar{\mathcal{G}}^k$ acts (infinitesimally) transitively on $\hat{\mathcal{M}}_{\mathcal{G},k}$. This action preserves the fibers of the projection $\hat{\mathcal{M}}_{\mathcal{G},k}\to\mathcal{M}_{\mathcal{G},k}$, and therefore, defines an action of $\bar{\mathcal{G}}^k$ on $\mathcal{M}_{\mathcal{G},k}$. The superalgebra $\mathcal{G}$ acts on $\mathcal{M}_{\mathcal{G},k}$ through the diagonal embedding $\mathcal{G}\hookrightarrow\bar{\mathcal{G}}^k$.*

6.2. Semigroups of $\mathcal{G}$-annuli

One may ask if it is possible to integrate the infinitesimal action of the superconformal algebra $\mathcal{G}$ on the moduli superspace $\mathcal{M}_{\mathcal{G},k}$ to a Lie group action. The answer is negative for a trivial reason: there is no group to act, since the Witt subalgebra of $\mathcal{G}$ does not correspond to a complex Lie group—there is no complexification of the group $Diff_+(S^1)$ (cf. [N]).

However, there exists a complex infinite-dimensional semigroup $\mathcal{A}$ constructed by Yu. Neretin [N] and G. Segal [S], which can be used instead of the nonexistent complexification of the group $Diff_+(S^1)$. An element of $\mathcal{A}$ is an equivalence class of Riemann surfaces which are topological annuli, and are equipped with parametrizations of their boudaries. The operation in $\mathcal{A}$ is defined by sewing boundary components. The semigroup $\mathcal{A}$ is a bounded domain, and $Diff_+(S^1)$ is its Shilov boundary. There exists a natural action of $\mathcal{A}$ on the Riemann surfaces with parametrizied boundary components which extends the Witt algebra action on the moduli space.

This construction can be generalized for the superconformal algebras and supercurves. In [V4] the semigroups of $\mathcal{G}$-annuli are defined, and their relations with the moduli of $\mathcal{G}$-curves are studied.

7. Concluding remarks

In the end I would like to mention several natural questions and problems. Some of them I hope to address in future publications.

(1) *Central extensions and determinant bundles.* The action of the Witt algebra on moduli spaces can be lifted to a canonical action of the Virasoro algebra on the total space of the determinant line bundle λ. The line λ_C at the point corresponding to the Riemann surface C is

$$\det H^0(C;\Omega_C)\otimes\det H^1(C;\Omega_C)^*, \tag{11}$$

where Ω_C is the sheaf of holomorphic one-forms on C.

The Lie algebra cocycle, corresponding to the extension $Vir\to W$, is induced under the action from the Chern class of the determinant bundle λ. A natural conjecture is that for any distinguished superconformal algebra $\mathcal{G}$, its central extension acts on a non-trivial line bundle over the $\mathcal{G}$-moduli space.

(2) *Mumford formula.* The line bundles λ_j on the moduli space of Riemann surfaces are defined by the formula (11) with Ω_C replaced by Ω_C^j. The Mumford isomorphism $\lambda_j\simeq\lambda_1^{6j^2-6j+1}$ is important in the bosonic string theory. It can be understood in terms of representations of the Virasoro algebra [ACKP]. What is an analogue of this formula for $\mathcal{G}$-supercurves?

For $K(1|1)$-supercurves (N = 1-super Riemann surfaces) the Mumford sheaves Λ_j were defined in [M1] and the formula $\Lambda_j=\Lambda_1^{2j-1}$ was proved by P. Deligne [D2] and A. Voronov [Vo].

(3) Find a relation between integrable positive energy representations of a superconformal algebra $\mathcal{G}$ and holomorphic representations of the semigroup of $\mathcal{G}$-annuli. For $\mathcal{G}=K(1|1)$ this is done in [V4].

(4) *Super KP hierarchies.* The Witt and Virasoro Lie algebras are also related with moduli spaces of algebraic curves via the Sato–Segal–Wilson ∞-dimensional Grassmannian and the Krichever construction in the theory of the Kadomtsev–Petviashvili hierarchy [SW]. What are the corresponding objects for superconformal algebras? Similar relations for $K(1|1)$ and $W(1|1)$ are found in [Rd, Rb, MR, Mu].

(5) *Compactifications.* Construct a compactification of the moduli superspace of G-curves. Even the compactification of the underlying manifold is a non-trivial and interesting problem.

For $\mathcal{G}=K(1|1)$ the even part of this superspace would be a compactification of the moduli space of algebraic curves with theta-characteristics. A compactification of this space was constructed by Cornalba [Co]. A compactification of the supermoduli space of $K(1|1)$-curves in the spirit of Deligne–Mumford was considered by Deligne [D1].

The even part of the moduli superspace of the untwisted $K(1|2)$-curves is just the universal Jacobian bundle over the moduli space of algebraic curves. The first work where a compactification of this space is considered is the thesis of L. Caporaso [Ca].

(6) *Almost complex $\mathcal{G}$-structures.* A $\mathcal{G}$-supercurve can be considered as a real super-

manifold of dimension $2|N$ with an extra structure. What are the corresponding torsion constraints? For supercurves of types W, $K(1|1)$, and $K(1|2)$ the integrability theorems are proved in [V2, LR, GNW].

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